ED-312

M.A./M.Sc. 1st Semester
Examination, March-April 2021

MATHEMATICS

Paper - IV

Advanced Complex Analysis-I

Time : Three Hours] [*Maximum Marks* : 80
[*Minimum Pass Marks* : 16

Note : Answer any **two** parts from each question. All questions carry equal marks.

Unit-I

1. (*a*) Let $f(z)$ be analytic within and on the boundary C of a simply connected region D and let a be any point within C. Then show that

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

(*b*) State and prove Liouville's theorem.

(*c*) Find the nature and location of the singularities of the function $f(z)=\dfrac{1}{z\left(e^{z}-1\right)}$. Prove that $f(z)$ can be expanded in the form $\dfrac{1}{z^{2}}-\dfrac{1}{2z}+a_{0}+a_{1}z^{2}+a_{4}z^{4}+\ldots$ where $0<|z|<2\pi$ and find the values of a_0 and a_2.

Unit-II

2. (*a*) If M is maximum value of $|f(z)|$ on and within C, then show that $|f(z)|<M$ for every point z within C, unless f is a constant.

(*b*) State and prove Schwarz's lemma.

(*c*) If $a>e$, use Rouche's theorem to prove that the equation $e^{z}=az^{n}$ has n roots inside the circle $|z|=1$.

Unit-III

3. (*a*) Show that

$$\int_{0}^{2\pi}\frac{d\theta}{a+b\cos\theta}=\frac{2\pi}{\sqrt{a^{2}-b^{2}}}$$

where $a>b>0$.

(*b*) Evaluate :

$$\int_0^\infty \frac{dx}{x^4 + a^4} \quad (a > 0)$$

(*c*) If $a > 0$, $m > 0$, prove that :

$$\int_0^\infty \frac{\cos mx}{\left(x^2 + a^2\right)^2}\, dx = \frac{\pi}{4a^3}(1 + ma)e^{-ma}$$

Unit-IV

4. (*a*) Let $f(z)$ be an analytic function of z in a domain D of the z-plane and let $f'(z) \neq 0$ inside D. Then show that the mapping $w = f(z)$ is conformal at all points of D.

(*b*) Show that the transformation $w = \left(\frac{z - ic}{z + ic}\right)^2$ where c is real, maps the right half of the circle $|z| = c$ into the upper half of the w-plane.

(*c*) Find all the mobius transformation which transform the unit circle $|z| \leq 1$ on to the unit circular disc $|w| \leq 1$.

Unit-V

5. (*a*) Let the metric ρ be defined by

$$\rho(f,g)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n} \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \quad \text{if } \epsilon > 0$$

is given then show that there is a $\delta > 0$ and a compact set $K \subset G$ such that for $f, g \in C(G, \Omega)$,

$\sup\{d(f(z), g(z) : z \in K\} < \delta \Rightarrow \rho(f, g) < \epsilon$

conversely if $\delta > 0$ and a compact set K are given then show that there is an $\epsilon > 0$ such that for $f, g \in (\subset, \Omega)$,

$\rho(f, g) < \epsilon \Rightarrow \sup\{d(f(z), g(z)) : z \in K\} < \delta.$

(*b*) State and prove Open mapping theorem.

(*c*) Let Ω be a simply connected region in the z-plane which is neither the z-plane itself nor the extended z-plane and let $z_0 \in \Omega$ then show that there is a unique analytic function $f : \Omega \to \not\subset$ having the properties :

(*i*) $f(z_0) = 0$ and $f'(z_0) > 0$;

(*ii*) f is one-one;

(*iii*) $w = f(z)$ maps Ω onto the disc $|w| < 1$.