FD-309

M.A./M.Sc. 1st Semester
Examination, Dec.-Jan., 2021-22

MATHEMATICS

Paper - I

Advanced Abstract Algebra - I

Time : Three Hours] [*Maximum Marks* : 80

Note : Answer any **two** parts from each question. All questions carry equal marks.

1. (*a*) Prove that any two composition series of a finite group are equivalent.

(*b*) Prove that every subgroup of a solvable group is solvable.

(*c*) If G be a nilpotent group, then prove that every subgroup of G and every homomorphic image of G are nilpotent.

2. (*a*) Let $F \leq E \leq K$ be fields. If K is a finite extension of E and E is a finite extension of F, then prove that K is a finite extension of F and

$$[K : F] = [K : E]\,[E : F].$$

(*b*) Prove that, a polynomial of degree n over a field can have at the most n roots in any extension field.

(*c*) Prove that, let F be a field. Then there exists an algebraically closed field K containing F as a subfield .

3. (*a*) Find the degree of splitting field $x^5 - 3x^3 + x^2 - 3$ over Q.

(*b*) Prove that, the prime field of a field F is either isomorphic to Q or to $z/(p)$, p is prime.

(*c*) If α, β be the algebric elements over a field F of characteristic zero, then prove that $F(\alpha, \beta)$ is a simple extension of F.

4. (*a*) Prove that, the set Aut (K) of all automorphisms of a field K form a group under composition of mappings.

(*b*) State and prove fundamental theorem of Galois theory.

(*c*) Find the Galois group of $x^3 - 2 \in Q[x]$.

5. (*a*) Prove that, any quartic over F is solvable by radicals.

(*b*) Show that the polynomial $2x^5 - 5x^4 + 5$ is not solvable by radicals.

(*c*) Let E be the splitting field of $x^n - a \in F(x)$. Then prove that $G(E/F)$ is a solvable group.

FD-310

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MATHEMATICS

Paper - II

Real Analysis - I

Time : Three Hours] [*Maximum Marks* : 80
[*Minimum Pass Marks* : 16

Note : Answer any **two** parts from each question. All questions carry equal marks.

Unit-I

1. (*a*) State and prove Cauchy's general principle of uniform convergences.

(*b*) Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in R(\alpha)$ on $[a, b]$ for $n = 1, 2, 3, ...$ and suppose $f_n \to f$ uniformly on $[a, b]$. Then prove that $f \in R(\alpha)$ on $[a, b]$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$.

(*c*) Show that the series $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...$ converges uniformly in $0 < a \le x \le b < 2\pi$.

Unit-II

2. (*a*) If $\sum a_n$ is a series of conplex number which converges absolutely, then prove that every rearrangement of $\sum a_n$ converges and they all converges to the same sum.

(*b*) State and prove the converse of Abel's theorem.

(*c*) Find the radius of convergence of the following series :

(*i*) $1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + ...$

(*ii*) $1 + 2x + 3x^2 + 4x^3 + ...$

Unit-III

3. (*a*) Let Ω be the set of all invertible linear operators on R^n.

(*i*) If $A \in \Omega$, $B \in L\,(R^n)$ and $\| B - A \| \, \| A^{-1} \| < 1$, then prove that $B \in \Omega$.

(*ii*) Ω is open subset. Is $L(R^n)$ and mapping $f : \Omega \to \Omega$ defined by $f(A) = A^{-1}$ for all $A \in \Omega$ is continuous ?

(*b*) State and prove the Taylor's theorem.

(*c*) State and prove the chain rule.

Unit-IV

4. (*a*) Determine the maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 + \frac{3\sqrt{3}}{2}xy$$

subject to the constraint $4x^2 + y^2 = 1$.

(*b*) Prove that of all rectangular parallel-opipeds of the same volume the cube has the least surface.

(*c*) If $u = \dfrac{x+y}{z}, v = \dfrac{y+z}{x}, w = \dfrac{y(x+y+z)}{xz}$,

show that u, v, w are not independent and find the relations among them.

Unit-V

5. (*a*) Write the definition of the following :

(*i*) The integral of 1-form

(*ii*) The integral of 2-form

(*iii*) The Triple integral

(*b*) State and prove the partitions of unity.

(*c*) Write a short note on differential form.

FD-311

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MATHEMATICS

Paper - III

Topology-I

Time : Three Hours] [*Maximum Marks* : 80

Note : Answer any **two** parts from each question. All questions carry equal marks.

Unit-I

1. (*a*) State and prove Schroeder-Bernstein theorem.

   (*b*) Give an example of a topological space different from the discrete and indiscrete spaces in which open sets are exactly the same as closed sets.

   (*c*) Let X be a topological space, and let $A \subset X$. Then prove that A is closed iff $D(A) \subset A$.

Unit-II

2. (*a*) Let X be any set and let $\wp$ be the Kuratowski closure operator on X. Then prove that there exists a unique topology τ on X such that for each $A \subset X$, $\wp(A)$ coincides with τ-closure of A.

(*b*) Prove that homeomorphism is an equivalence relation in the collection of all topological spaces.

(*c*) Prove that the property of being a Lindelöf space is a topological property.

Unit-III

3. (*a*) Prove that every subspace of a Hausdorff space is Hausdorff.

(*b*) Prove that a topological space X is normal iff for any closed set F and open set G containing F, there exists an open set V such that

$$F \subset V \subset \overline{V} \subset G$$

(*c*) State and prove Tietze's extension theorem.

Unit-IV

4. (*a*) Prove that a subset A of R is compact iff A is closed and bounded.

(*b*) Prove that a topological space is countably compact iff every countable collection of closed subsets of X with FIP has non-empty intersection.

(*c*) Let (X^*, τ^*) be a one point compactification of a non-compact topological space (X, τ). Then prove that (X^*, τ^*) is Hausdorff iff (X, τ) is Hausdorff and locally compact.

Unit-V

5. (*a*) Let (X, d) be a metric space. Then prove that the following statements are equivalent :

(*i*) X is compact

(*ii*) X is countably compact

(*iii*) X has BWP

(*iv*) X is sequentially compact

(*b*) Prove that a topological space X is disconnected iff there exists a non-empty proper subset of X which is both open and closed in X.

(*c*) Prove that every component of a locally connected space is an open set.

FD-312

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MATHEMATICS

Paper - IV

Advanced Complex Analysis-I

Time : Three Hours] [*Maximum Marks* : 80
[*Minimum Pass Marks* : 16

Note : Answer any **two** parts from each questions. All questions carry equal marks.

Unit-I

1. (*a*) State and prove Cauchy's Integral formula.

(*b*) Prove that the function $\sin\left[c\left(z+\frac{1}{z}\right)\right]$ can be expanded in a series of the types $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ in which the coefficients of both z^n and z^{-n} are :

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\left[\sin\left(2c\cos\theta\right)\cos n\theta\right]}{\sin\left(2c\cos\theta\right)\cos n\theta}\,d\theta$$

(*c*) Define Entire function. Find the singularity of the function $\dfrac{e^{c/(z-a)}}{e^{z/a}-1}$, indicating the characters of each singularity.

Unit-II

2. (*a*) Prove that all the roots of $x^7 - 5z^3 + 12 = 0$ between the circles $|z| = 1$ and $|z| = 2$.

(*b*) State and prove maximum modulus principle.

(*c*) State and prove Inverse function theorem.

Unit-III

3. (*a*) Apply the calculus of residue to prove that :

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

(*b*) Show that :

$$\int_0^\pi \frac{\cos 2\theta}{1-2a\cos\theta + a^2}\, d\theta = \frac{\pi a^2}{1-a^2}\left(a^2 < 1\right)$$

(*c*) Prove by contour integration :

$$\int_0^\infty \frac{\log\left(1+x^2\right)}{1+x^2}\, dx = \pi \log 2$$

Unit-IV

4. (*a*) Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$ and explain why the curve obtained is not a circle.

(*b*) Let $f(z)$ be analytic function of z in a region D of the z-plane and $f'(z) \neq 0$ inside D. Then prove that the mapping $w = f(z)$ is conformal at the point of D.

(*c*) Discuss the transformation $w = \tan z$.

Unit-V

5. (*a*) Show that, A family F of holomorphic function defined in a domain D, that is $F \subset H(D)$ is normal iff F is locally bounded.

(*b*) Let $\{f_n\}$ be a sequence in $H(G)$ and $f \in (G, C)$ such that $f_n \to f$. Then show that f is analytic and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$.

(*c*) State and prove Riemann mapping theorem.

FD-313

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MATHEMATICS

Paper - V

Advanced Discrete Mathematics - I

Time : Three Hours] [*Maximum Marks* : 80
[*Minimum Pass Marks* : 16

Note : Answer any **two** parts from each question. All questions carry equal marks.

Unit-I

1. (*a*) Define Tautology. If $H_1, H_2, ..., H_m$ and P imply Q, then prove that $H_1, H_2, ..., H_m$ imply $P \to Q$.

(*b*) What are the quantifiers ? Explain universal quantifiers and existential quantifiers.

(*c*) Show that :

$$\rceil(P \wedge Q) \to \left(\rceil P \vee \left(\rceil P \vee Q\right)\right) \Leftrightarrow \left(\rceil P \vee Q\right)$$

Unit-II

2. (*a*) State and prove basic homomorphism theorem.

(*b*) Define submonoid and prove that for any commutative monoid (M, *) the set of idempotent elements of M forms a submonoid.

(*c*) Define direct product of semigroup. Show that the direct product of any two semigroups is a semigroup.

Unit-III

3. (*a*) Define Distributive lattice. Let (L, *, ⊕) be a distributive lattice. For any $a, b, c \in L$, prove that

$$(a * b = a * c) \wedge (a \oplus b = a \oplus c) \Rightarrow b = c$$

(*b*) State and prove De Morgan's law.

(*c*) Define the following

(*i*) Sublattice

(*ii*) Direct product

(*iii*) Boolean algebra

(*iii*) Lattice as partially order set

Unit-IV

4. (*a*) Use the Karnaugh map representation to find a minimal sum-of-product of the following function :

$$f(a, b, c, d) = \Sigma(10, 12, 13, 14, 15)$$

(*b*) Define the following :

(*i*) Join-irreducible

(*ii*) Atoms and Minterms

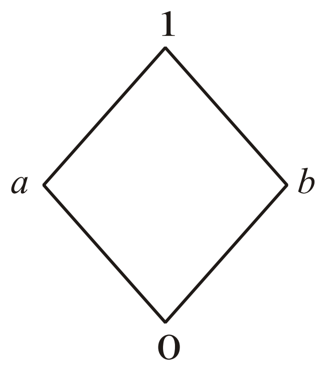
(*iii*) Gates

(*iv*) Canonical forms

(*c*) Find the value of

$$x_1 * x_2 \left[(x_1 * x_4) \oplus x_2' \oplus (x_3 * x_1') \right]$$

for $x_1 = a$, $x_2 = 1$, $x_3 = b$ and $x_4 = 1$, where $a, b, 1 \in B$ and the Boolean algebra $(B, *, \oplus, ', 0, 1)$ is shown in the following figure :



Unit-V

5. (*a*) Define Polish notation; prove that the rank of any well formed polish formula is 1 and the rank of any proper head of a polish is greater than or equal to 1.

(*b*) State and prove Pumping Lemma.

(*c*) Define grammar and let language

$L(Gs) = \left\{ a^n b^n c^n \mid n \geq 1 \right\}$ is generated by the following grammar

$$Gs = \left\langle \{S, B, C\}, \{a, b, c\} S, \phi \right\rangle$$

where ϕ consists of the production
$S \to aSBC$, $S \to aBC$, $CB \to BC$,
$aB \to ab$, $bB \to bb$ $bC \to bc$, $cC \to cc$
then find the derivation for the strings abc, $a^2b^2c^2$ and $a^3b^3c^3$.